

# Regularity Criteria of BKM type in Distributional Spaces for the 3-D Navier-Stokes Equations on Bounded Domains

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## Abstract

In the classic work of Beale-Kato-Majda ([2]) for the Euler equations in  $\mathbb{R}^3$ , regularity of a solution throughout a given interval  $[0, T_*]$  is obtained provided that the curl  $\omega$  satisfies  $\omega \in L^1((0, T); L^\infty(\mathbb{R}^3))$  for all  $T < T_*$ , and the authors noted that the arguments apply equally well to the Navier-Stokes equations (NSE) in  $\mathbb{R}^3$ . In later works by various authors the spatial  $L^\infty$ -criterion imposed on the curl was generalized to a *BMO* criterion, and later to a Besov space criterion, in both the Euler and NSE cases ([9], [10], [11]). Meanwhile, the authors in [2] remarked that additional ideas seem necessary to obtain results of this type on bounded spatial domains. Efforts in this direction in [8] for the NSE case produced regularity results with the *BMO* criterion imposed on localized balls.

In this paper for the NSE case and on general bounded domains  $\Omega$  in  $\mathbb{R}^3$ , we obtain a regularity result of BKM type that goes beyond function spaces to spatially allow  $\omega$  to be a distribution. This is done by making a new connection between a well-known vector calculus result and the classical regularity criteria of Serrin type ([12], [14], [15], [18]). Specifically, for certain Sobolev spaces  $H^{s,p}(\Omega)$  suitably defined for  $s < 0$  we show that if  $u$  is a Leray solution of the 3-D NSE on the interval  $(0, T)$  and if  $\omega \in L^s((0, T); H^{-1,p}(\Omega))$  where  $\frac{2}{s} + \frac{3}{p} = 1$  for some  $p \in (3, \infty]$ , then  $u$  is a regular solution on  $(0, T]$ ; in particular for  $p = \infty$  we have a regular solution when  $\omega \in L^2((0, T); H^{-1,\infty}(\Omega))$ , which directly strengthens the results

in [2] by one order of (negative) derivative in terms of the spatial criteria for regularity. Our results thus impose more stringent conditions on time than the BKM results and their generalizations described above, but as far as we are aware the results here represent the first of BKM type for the NSE that allow  $\omega$  to spatially be a distribution.

**Keywords:** BKM criteria, curl, regularity, vector-calculus identity, duality arguments.

## 1 Introduction

We consider the 3-D Navier-Stokes equations for viscous incompressible homogeneous flow

$$u_t + \nu Au + (u \cdot \nabla) u + \nabla p = g, \quad (1.1a)$$

$$\nabla \bullet u = 0. \quad (1.1b)$$

Here  $\Omega$  is a bounded spatial domain in  $\mathbb{R}^3$  with sufficiently smooth boundary and  $u = (u_1, u_2, u_3)$  with  $u_i = u_i(x, t)$ ,  $x \in \Omega$ ,  $1 \leq i \leq n$  and  $t \geq 0$ . The external force is  $g = (g_1, g_2, g_3)$ , with  $g_i = g_i(x, t)$ , and  $p = p(x, t)$  is the pressure. The domain  $\Omega$  can be either a periodic box or a Lipschitz domain with zero (no-slip) boundary conditions; in the latter case, or by "moding out" the constant vectors as in standard practice in the former case,  $A = -\Delta$  has eigenvalues  $0 < \lambda_1 < \lambda_2 < \dots$  with corresponding eigenspaces  $E_1, E_2, \dots$ , so that in particular  $A$  is a positive definite operator and  $A^{-1}$  is a well-defined bounded operator on the Banach spaces  $L^p(\Omega)$ ,  $p \in [1, \infty)$ . Let  $\partial_x$  denote the operator  $\frac{\partial}{\partial x}$  then with similar definitions for  $\partial_y$  and  $\partial_z$  we have that  $\nabla \bullet u = \text{div } u = \partial_x u_1 + \partial_y u_2 + \partial_z u_3$ . Of particular interest also is the curl  $\omega$  defined by  $\omega = \nabla \times u = (\partial_y u_3 - \partial_z u_2, \partial_z u_1 - \partial_x u_3, \partial_x u_2 - \partial_y u_1)$ . With zero viscosity ( $\nu = 0$ ) the system (1.1) becomes the Euler system

$$u_t + (u \cdot \nabla) u + \nabla p = 0, \quad (1.2a)$$

$$\nabla \bullet u = 0. \quad (1.2b)$$

In the classical work of Beale/Kato/Majda ([2]),  $\Omega = \mathbb{R}^3$ ,  $g = 0$ , and regularity for a smooth solution of (1.2) throughout a given interval  $[0, T_*]$  is obtained provided that  $\omega \in L^1((0, T); L^\infty(\mathbb{R}^3))$  for all  $T < T_*$ . Central to the arguments in [2] is the formula  $u = -\nabla \times (\nabla^{-1} \omega)$  which in  $\mathbb{R}^3$  is given explicitly by appropriately available kernels via the Biot-Savart Law. The authors note that the results hold for periodic flow with minor modification, and they note that the results apply to the NSE as well.

The results in [2] for (1.1) were extended in the case  $\Omega = \mathbb{R}^n$  in [9] to allow  $\omega \in L^2((0, T); BMO)$  where  $BMO$  denotes the class of functions of bounded mean oscillation. Later in [10] this condition was extended to both (1.1) and (1.2) to allow  $\omega \in L^1((0, T); BMO)$ . Then in [11] the results in [10] were extended to allow  $BMO$  to be replaced by the Besov space  $B_{\infty, \infty}^0$ . The same regularity criterion developed in [11] was then established in the case  $n = 3$  for the Boussinesq system, the MHD system, and a fluid system with the linear Soret effect in [4], [13], and [5], respectively. Meanwhile the authors of [2] noted that a more involved proof using additional ideas seems necessary for bounded spatial domains. In [8] regularity results were obtained for the NSE case by imposing the  $BMO$  condition on localized balls.

In this paper for the NSE case and on general bounded domains  $\Omega$  in  $\mathbb{R}^3$  with sufficiently smooth boundary we will obtain regularity results of BKM type in which the spatial criteria that we impose on  $\omega$  will allow  $\omega$  to lie in negative Sobolev spaces. Thus a.e. for each  $t$  the curl  $\omega(\bullet, t)$  is allowed to be a distribution.

In proving our results we will make use of the classic regularity criteria for the Navier-Stokes equations which establish regularity of Leray solutions (see the definition in section 2 below) provided that  $u \in L^\theta((0, T); L^p(\Omega))$  and  $\theta, p, n$  satisfy  $\frac{2}{\theta} + \frac{n}{p} = 2$ ,  $n < p \leq \infty$ . Here  $\Omega = \mathbb{R}^n$  or under suitable conditions such as those assumed here  $\Omega$  is a bounded domain; see [12], [14], [15], [18], and the references contained therein. Preliminary results toward extending these classic results to the borderline case  $n = p$  were obtained in [6], [7], [19], [20] (see also the references contained therein), and recently this borderline result was obtained in the case  $\Omega = \mathbb{R}^3$  ([3], [16]). It is as yet unknown if the borderline case can be obtained on bounded domains.

The other main component used in establishing our results will be the well-known vector-calculus identity

$$Av = \nabla \times \nabla \times v \tag{1.3}$$

holding for smooth divergence-free vector fields on  $\Omega$ . The smoothness we require for the boundary of  $\Omega$  is that the usual Sobolev inequalities hold. Since on  $\Omega$  under these conditions and for the assumed boundary conditions (e.g. zero Dirichlet) the operator  $A$  is invertible, we have from (1.3) that  $v = A^{-1}(\nabla \times \nabla \times v)$  which provides an alternative relationship

between  $u$  and  $\omega$  similar to  $u = -\nabla \times (\nabla^{-1}\omega)$  but more adaptable to bounded domains and more directly applicable to our techniques. The following result easily generalizes the identity  $v = A^{-1}(\nabla \times \nabla \times v)$ :

**Theorem 1** *If  $u$  is a smooth enough solution of (1.1) or (1.2) then  $A^s u = A^{s-1}(\nabla \times \omega)$ , where  $s$  is any order allowed by the smoothness of  $u$ .*

Here as in standard fashion we let  $H_0 \equiv \overline{\{v \in C_0^\infty(\Omega) : \nabla \bullet v = 0\}}_{L^2(\Omega)}$ , ie.  $H_0$  is the closure in  $L^2(\Omega)$  of the smooth compactly-supported solenoidal vector fields. Note that  $v$  need not be smooth in order for the relationship  $v = A^{-1}(\nabla \times \nabla \times v)$  to hold, since  $A^{-1}(\nabla \times \nabla \times \bullet)$  defines a bounded operator on  $H_0$  as can bwe quickly seen (see section 2 below).

Recall that the standard Sobolev spaces  $W^{k,p}(\Omega)$  are defined as  $W^{k,p}(\Omega) \equiv \{v \in L^p(\Omega) \mid D^\alpha v \in L^p(\Omega) \forall |\alpha| \leq k\}$ . The corresponding negative Sobolev spaces are defined for each  $k$  as the dual spaces of  $W^{k,p'}(\Omega)$  where  $\frac{1}{p} + \frac{1}{p'} = 1$ , i.e.  $W^{-k,p}(\Omega) \equiv (W^{k,p'}(\Omega))'$ . Our use of Sobolev spaces of negative or positive order is motivated by the following characterization (see e.g. [17]):

**Theorem 2** *Let  $v \in \mathcal{D}'(\Omega)$ , then  $v \in W^{-k,p}(\Omega)$  if and only if  $v = \sum_{|\alpha| \leq k} D^\alpha w_\alpha$  where  $w_\alpha \in L^p(\Omega)$ .*

Theorem 2 says in a sense that  $v \in W^{-k,p}(\Omega)$  iff  $D^{-k}v \in L^p(\Omega)$ ; we can make this more precise by defining suitable Sobolev spaces  $H^{-s,p}(\Omega)$  for any  $s \geq 0$  by  $H^{-s,p}(\Omega) \equiv \{v \in \mathcal{D}'(\Omega) \mid A^{-s/2}v \in L^p(\Omega)\}$  where as noted we have assumed that  $\Omega$  has a boundary smooth enough such that the usual Sobolev spaces as well as the operators  $A^{-s/2}$  are well-defined. In fact, if we take this together with the definition  $H^{s,p}(\Omega) \equiv \{v \in D(A^{s/2})\}$  with norm  $\|v\|_{s,p} \equiv \|A^{s/2}v\|_p$  then we have a consistent definition of  $H^{s,p}(\Omega)$  for any real  $s$  and any  $p \in (1, \infty]$ . These are the Sobolev spaces we will work with, of both negative and positive order. Similar spaces were defined and used in [7] and [20] wherein  $A$  in those cases was the Stokes operator  $-P\Delta$  where  $P$  is the Leray projection onto the solenoidal vectors. From the basic tools developed in Theorem 1, the regularity criteria  $u \in L^\theta((0, T); L^p(\Omega), \frac{2}{\theta} + \frac{3}{p} = 2, 3 < p \leq \infty$  as noted above for the case  $n = 3$ , and our definition here of the spaces  $H^{-s,p}(\Omega)$  we will establish the main result of this paper:

**Theorem 3** *Let  $u$  be a Leray solution of (1.1) on the interval  $(0, T)$  and suppose that  $\omega \in L^\theta((0, T); H^{-1,p}(\Omega))$  where  $\frac{2}{\theta} + \frac{3}{p} = 1$  for some  $p \in (3, \infty]$ . Then  $u$  can be continued to a regular solution of (1.1) on  $(0, T]$ .*

We remark that as an immediate corollary of Theorem 3 (and as noted similarly in [2], [9], [10], [11]) we have that if the maximal existence time  $T^*$  is finite then  $\limsup_{t \uparrow T^*} \|\omega(t)\|_{H^{-1,\infty}(\Omega)} = \infty$ . Theorem 3 overlaps with the main result of [2] and the results in [4], [5], [10], [11], [13] in that the condition on the integrability in time is more stringent while the spatial requirement on  $\omega$  is more general. Specific to the case  $p = \infty$  the results in [2] require that  $\omega \in L^1((0, T); L^\infty(\Omega))$  whereas here the corresponding condition is that  $\omega \in L^2((0, T); H^{-1,\infty}(\Omega))$ ; this means in particular that  $A^{-1/2}\omega(\bullet, t) \in L^\infty(\Omega)$  a.e. for each  $t$  in contrast with the requirements in [2] which imply that  $\omega(\bullet, t) \in L^\infty(\Omega)$  a.e. for each  $t$ . Theorem 3 will follow by connecting the results of Theorem 1 with the regularity criteria  $u \in L^\theta((0, T); L^p(\Omega))$ ,  $\frac{2}{\theta} + \frac{3}{p} = 2$ ,  $3 < p \leq \infty$  by using a few duality arguments similar to those employed in [7] and in [1]; after some preliminary discussion Theorems 1 & 3 will be proven in the next section. In section 3 we will make some concluding remarks and observations.

## 2 Preliminaries and Proof of Theorem 3

By a Leray solution of (1.1) on  $(0, T)$  we mean a vector  $u \in L^\infty((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$  satisfying, for each  $v \in L^\infty((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$ , the equation

$$(u(t), v) + \nu \int_{t_0}^t (A^{1/2}u, A^{1/2}v) + ((u \cdot \nabla)u, v) ds = (u(t_0), v) + \int_{t_0}^t (g, v) ds \quad (2.1)$$

for all intervals  $(t_0, t)$  contained in  $(0, T)$ . Since  $((u \cdot \nabla)u, uv) = -((\nabla \bullet u)u, u) = 0$  and by the standard use of Young's inequality on the term  $(u(t_0), u)$  we have by setting  $v = u$  in (2.1) that

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_{t_0}^t \|A^{1/2}u\|_2^2 ds \leq \frac{1}{2} \|u(t_0)\|_2^2 + \int_{t_0}^t (g, u) ds. \quad (2.2)$$

and hence Leray solutions  $u$  also satisfy the standard energy inequality. Such solutions that also satisfy one of the criteria  $u \in L^\theta((0, T); L^p(\Omega))$ ,  $\frac{2}{\theta} + \frac{3}{p} = 2$ ,  $3 < p \leq \infty$  are in fact regular solutions of (1.1) on  $(0, T)$  by the classic regularity results mentioned above in the introduction.

We recall that smooth vector fields  $v$  which vanish on  $\partial\Omega$  in the sense of weak solutions of the Laplace equation satisfy  $Av + \nabla(\nabla \bullet v) = \nabla \times (\nabla \times v)$ , and thus if  $v$  is divergence-free, i.e.  $\nabla \bullet v = 0$ , then we have the well-known result that

$$Av = \nabla \times \nabla \times v \quad (2.3)$$

as noted in the introduction. Hence, since under the assumed (e.g. zero Dirichlet) boundary conditions  $A$  is positive definite and has a well-defined bounded inverse  $A^{-1}$  (with  $\|A^{-1}\|_2 = \lambda_1^{-1}$ ),

$$v = A^{-1}(\nabla \times \nabla \times v) \quad (2.4)$$

and by applying  $A^s$  to both sides we obtain Theorem 1 for suitably smooth  $v$ . Note that (2.3) holds also in the distributional sense by considering the application of the appropriate adjoint operators to smooth test functions; hence (2.4) can hold in this sense for nonsmooth  $v$  as well. In fact  $B_0 \equiv A^{-1}(\nabla \times \nabla \times)$  defines a bounded operator on  $H_0$ . For  $D_c \equiv \nabla \times$  we have that  $B_0^* = (D_c^*)^2 A^{-1}$  is clearly a bounded operator on  $H_0$  so the result follows by duality; similar arguments were employed in [7] to show that the operator  $A^{-1/2} P \operatorname{div}$  is a bounded operator from  $L^p(\Omega)$  to  $PL^p(\Omega)$ , and in [1] for a related class of operators and spaces.

We begin the proof of Theorem 3 by setting  $B_1 \equiv A^{-1/2} D_c$ ; by duality again  $B_1$  is a bounded operator on  $L^p(\Omega)$ ,  $1 < p < \infty$ . Then for  $u \in L^p(\Omega)$  we have that  $A^{s/2} B_1 u = A^{s/2} (A^{-1/2} D_c A^{-s/2}) A^{s/2} u = [A^{(s-1)/2} D_c A^{-1/2} A^{(1-s)/2}] A^{s/2} u$ . The operator  $B_2 \equiv D_c A^{-1/2}$  is clearly a bounded operator on  $L^p(\Omega)$ ,  $1 < p < \infty$ ; set  $B_3 \equiv A^{(s-1)/2} D_c A^{-1/2} A^{(1-s)/2} = A^{(s-1)/2} B_2 A^{(1-s)/2}$  then  $B_3$  is therefore also a bounded operator on  $L^p(\Omega)$ ,  $1 < p < \infty$ , directly if  $(1-s)/2 \leq 0$  and by duality again if otherwise. Thus  $A^{s/2} (B_1 u) = B_3 (A^{s/2} u)$  and so  $B_1 \equiv A^{-1/2} D_c$  is a bounded operator on  $H^{s,p}(\Omega)$  for any real  $s$  and any  $p \in (1, \infty)$ .

Then from (2.4) we have for any  $u \in L^p(\Omega)$  that  $u = A^{-1}(\nabla \times \nabla \times u) = A^{-1/2} (A^{-1/2} D_c)(\nabla \times u) = A^{-1/2} B_1 \omega$ . Since clearly  $A^{-1/2}$  is a bounded operator from  $H^{s-1,p}(\Omega)$  to  $H^{s,p}(\Omega)$  for any real  $s$  and any  $p \in (1, \infty)$ , we thus have in particular that if  $\omega \in H^{s-1,p}(\Omega)$  then  $u \in H^{s,p}(\Omega)$  for any real  $s$  and any  $p \in (1, \infty)$ ; setting  $s = 0$  and  $p \in (3, \infty)$  we thus obtain Theorem 3 for finite  $p$  since  $H^{0,p}(\Omega) = L^p(\Omega)$ . For the case  $p = \infty$  we observe that since we are on a bounded domain we have for any  $r \in [1, \infty)$  that  $\|A^{-1/2} \omega\|_r \leq |\Omega|^{1/r} \|A^{-1/2} \omega\|_\infty \leq$

$\|A^{-1/2}\omega\|_\infty$  if  $|\Omega| \leq 1$  and that  $\|A^{-1/2}\omega\|_r \leq |\Omega|^{1/r} \|A^{-1/2}\omega\|_\infty \leq |\Omega| \|A^{-1/2}\omega\|_\infty$  if  $|\Omega| \geq 1$ . Then combining with the remarks above we have that  $\|u\|_r$  is uniformly bounded by  $\|A^{-1/2}\omega\|_\infty$  for all  $r \in (1, \infty)$ , so since  $\lim_{r \rightarrow \infty} \|u\|_r = \|u\|_\infty$  we have that  $\|u\|_\infty$  is uniformly bounded by  $\|A^{-1/2}\omega\|_\infty$  and we thus obtain Theorem 3 for the case  $p = \infty$ .

### 3 Conclusion

On reasonable bounded domains with zero boundary conditions  $A^{-1}$  is well-defined, and with it we are able to replace the formula  $u = -\nabla \times (\nabla^{-1}\omega)$  and the use of the Biot-Savart Law with the identity (2.5). Duality arguments along the lines of those employed in [7] and in [1] then allow us to use this identity to connect with the standard regularity criteria  $u \in L^\theta((0, T); L^p(\Omega))$ ,  $\frac{2}{\theta} + \frac{3}{p} = 2, 3 < p \leq \infty$  via suitable operator-theory machinery. The identity (1.3) and the invertibility of  $A$  in fact are the key tools that allow us here from the outset to consider results of BKM type on bounded domains, and once in place we see that they allow us to take the extra step into distributional spaces.

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